

Note on three-generation models in heterotic string and F-theory on elliptic Calabi-Yau manifolds over Hirzebruch varieties

Shun'ya Mizoguchi* and Tomoki Sakaguchi†

**Theory Center, Institute of Particle and Nuclear Studies, KEK*

Tsukuba, Ibaraki, 305-0801, Japan and

**†SOKENDAI (The Graduate University for Advanced Studies)*

Tsukuba, Ibaraki, 305-0801, Japan

(Dated: July 7, 2016)

Abstract

We give a complete list of a class of three-generation models in $E_8 \times E_8$ heterotic string theory and its dual F-theory on an elliptic Calabi-Yau over a (generalized) Hirzebruch variety in which the divisors of the relevant line bundles needed for a smooth Weierstrass model are effective. The most stringent constraint on the bound of the eta class comes from the effectiveness of the divisor of the bundle corresponding to the highest Casimir in Looijenga's weighted projective space, as well as from the compactness of the toric variety. Comparison is also made with the list obtained in the literature.

PACS numbers:

* E-mail:mizoguch@post.kek.jp

† E-mail:stomoki@post.kek.jp

The origin of three generations of flavors in particle physics is an enigma. There has been no evidence found for the fourth new generation at the LHC experiment (see e.g.[1]), and the results of PLANCK has shown that the CMB data is consistent with the effective number of neutrinos derived by assuming three neutrino flavors (see e.g.[2]).

Although so far none of the string compactifications can explain why there are three generations in nature, F-theory/ $E_8 \times E_8$ heterotic models have the following advantages over other string (in particular D-brane) models: (1) They naturally lead to $SU(5)$ GUT models which beautifully explain the observed hypercharge assignments, which are otherwise difficult to explain. (2) They can realize the spinor representation of $SO(10)$, which can contain all the quarks and leptons in one generation as a complete multiplet. (3) They can give up-type Yukawa couplings, which would be perturbatively forbidden in D-brane models. (For a rather recent accumulation of literature on F-theory models, see the seminal papers [3–6] and their citations.)

In the standard heterotic string compactification to four dimensions with a hermitian vector bundle V [7], the number of chiral generation is given by, assuming $c_1(V) = 0$, a half the third Chern class $\frac{1}{2}c_3(V)$. Some time ago, it was found [8] that this was given, for a vector bundle characterized by the η class (the first Chern class of the twisting line bundle, corresponding to the number of instantons) and the γ class (the kernel of the projection from the spectral cover to the base which leads to some ambiguity of the vector bundle, corresponding to the G -flux in F-theory), as

$$\frac{1}{2}c_3(V) = \lambda\eta(\eta - nc_1) \tag{1}$$

for an $SU(n)$ bundle, where λ is a half-integral number related to the γ class, and c_1 is the first Chern class of the base.

In [8], a list of η classes leading to precisely three generations was also given, for $n = 5, 4$ and 3 corresponding to the $SU(5)$, $SO(10)$ and E_6 gauge group, respectively, for the base being a Hirzebruch surface or a del Pezzo surface. (See also [11], where the lower bound of the η class was also mentioned.) Not all the divisors in the list, however, turn out to be not usable, since the relevant divisors are not effective and holomorphic sections do not exist.

In this letter, we will solve the equation (1) in an elementary way for the case of the base being a Hirzebruch surface, for which the dual F-theory Calabi-Yau becomes an elliptic

fibration over a generalized Hirzebruch variety¹, to give a *complete* list of three-generation models with relevant effective divisors in this class of compactifications.

Of course, three-generation models in string theory are not rare,² nor does our result explain the mystery of the number of generations of quarks and leptons. We are, however, interested in these models because they are simple, and still infinitely many, so that they may serve as useful models to study some fundamental questions in F-theory model buildings, such as the multiple singularity enhancement and associated family unification [15, 16], computations of Yukawa couplings and nonperturbative superpotentials, mechanisms of moduli stabilization and supersymmetry breaking, etc. We plan to explore these issues elsewhere.

The Hirzebruch surface \mathbb{F}_m ($m = 0, 1, 2, \dots$) is a complex-dimension two manifold with coordinates $(z', w', z, w) \in \mathbb{C}^4$ with the identifications of points

$$(z', w', z, w) \sim (\mu z', \mu w', \kappa \mu^m z, \kappa w) \quad (2)$$

for arbitrary $\kappa, \mu \in \mathbb{C}^* = \mathbb{C} - \{0\}$, where the points satisfying $z = w = 0$ or $z' = w' = 0$ are removed. The weights of the identifications are summarized by the table

$$\begin{array}{ccccc} & z' & w' & z & w \\ \mu & 1 & 1 & m & 0 \\ \kappa & 0 & 0 & 1 & 1 \\ D_1 & D_2 & D_3 & D_4 \end{array}, \quad (3)$$

where we have also displayed the corresponding divisors in the bottom row. The associated toric fan is taken to be

$$\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & -m & 1 & -1 \end{array}, \quad (4)$$

which is nothing but the orthogonal complement of (3) if the numbers are viewed as a collection of row vectors. (4) implies the relations

$$\begin{aligned} D_1 &= D_2, \\ D_3 &= D_4 + mD_2. \end{aligned} \quad (5)$$

¹ See [9–11] for earlier works on F-theory on elliptic fourfolds over toric varieties.

² The literature on the construction of three-generation models is vast; a few of the notable papers include [12–14].

Writing $D_1 = D_2 = f$, $D_4 = C_0$ and $D_3 = C_\infty$, they are known to have intersections

$$C_0^2 = -m, \quad f^2 = 0, \quad C_0 \cdot f = 1. \quad (6)$$

The anticanonical class is

$$c_1(\mathbb{F}_m) = 2C_0 + (2 + m)f. \quad (7)$$

The generalized Hirzebruch variety \mathbb{F}_{mqp} ($m, q, p = 0, 1, 2, \dots$) we consider is a complex-dimension three manifold with coordinates $(z'', w'', z', w', z, w) \in \mathbb{C}^6$, which are similarly subject to the identifications

$$(z'', w'', z', w', z, w) \sim (\nu z'', \nu w'', \mu \nu^m z', \mu w', \kappa \mu^q \nu^p z, \kappa w) \quad (8)$$

for arbitrary $\kappa, \mu, \nu \in \mathbb{C}^*$, with the points satisfying $z = w = 0$ or $z' = w' = 0$ or $z'' = w'' = 0$ being all removed. The weights and divisors are

	z''	w''	z'	w'	z	w	
ν	1	1	m	0	p	0	
μ	0	0	1	1	q	0	.
κ	0	0	0	0	1	1	
	D_1	D_2	D_3	D_4	D_5	D_6	

(9)

The corresponding fan

1	-1	0	0	0	0
0	- m	1	-1	0	0
0	- p	0	- q	1	-1

(10)

also implies the relations

$$\begin{aligned} D_1 &= D_2, \\ D_3 &= D_4 + mD_2, \\ D_5 &= D_6 + pD_2 + qD_4, \end{aligned} \quad (11)$$

so that the anticanonical class is ($D_2 = f$, $D_4 = C_0$)

$$\begin{aligned} c_1(\mathbb{F}_{mqp}) &= (2 + q)C_0 + (2 + m + p)f + 2D_6 \\ &= c_1(\mathbb{F}_m) + (qC_0 + pf) + 2D_6. \end{aligned} \quad (12)$$

The “base part” of (12) differs from $c_1(\mathbb{F}_m)$ by $qC_0 + pf$ so that an elliptic fibration over \mathbb{F}_{mqp} can be used [11] for a dual description of the heterotic compactification on the elliptic fibration over \mathbb{F}_m whose the vector bundle is (partly) characterized by the class $\eta = 6c_1(\mathbb{F}_m) + qC_0 + pf$ of the twisting line bundle [7].

Let us now solve (1). Putting

$$\eta = xC_0 + yf \quad (x, y \in \mathbb{Z}), \quad (13)$$

(1) is equivalent to the equation

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x - 2n \\ y - (2 + m)n \end{pmatrix} = \frac{3}{\lambda}, \quad (14)$$

which can be solved for y as

$$y = \frac{m}{2}x + n + \frac{n^2 + \frac{3}{2\lambda}}{x - n}. \quad (15)$$

λ is taken to be $\pm\frac{1}{2}$ or $\pm\frac{3}{2}$, and only for $n = 4$, $\lambda = \pm 1, \pm 3$ is also allowed. Since y is an integer, the right hand side of (15) must also sum up to an integer. If $\lambda \in \mathbb{Z} + \frac{1}{2}$, This implies the following:

(I) If $mx \in 2\mathbb{Z}$, then the integer $n^2 + \frac{3}{2\lambda}$ must be divisible by $x - n$.

(II) If $mx \in 2\mathbb{Z} + 1$, then the integer $2(n^2 + \frac{3}{2\lambda})$ must be divisible by $x - n$, and at the same time the quotient must be an odd integer.

In addition, if $n = 4$ and $\lambda = \pm 1$ or ± 3 , then

(I') If $mx \in 2\mathbb{Z}$, then the integer $2(n^2 + \frac{3}{2\lambda})$ must be divisible by $x - n$, and at the same time the quotient must be an even integer.

(II') If $mx \in 2\mathbb{Z} + 1$, then the integer $2(n^2 + \frac{3}{2\lambda})$ must be divisible by $x - n$, and at the same time the quotient must be an odd integer.

Not all the solutions to the above are suitable for smooth compactifications. First of all, both η and $12c_1 - \eta$ must be effective (= all the coefficients of the divisor are non-negative), because they are the classes of the twisting line bundles of the spectral cover. The physical interpretation of this condition is that the instanton number of one of E_8 does not exceed

24 for each $K3$ fiber in the heterotic string compactification. In fact, the consideration of Looijenga's weighted projective space bundle [7, 17] provides a more stringent constraint on the possible η . Looijenga's weighted projective space is known as the moduli space of the vector bundle for the heterotic compactifications, and the section of the bundle consists of the independent polynomials in the coefficients of the Weierstrass model in F-theory. In the $SU(5)$ case, they are the polynomials

$$h_{\tilde{n}+2}, H_{\tilde{n}+4}, p_{\tilde{n}+6}, f_{\tilde{n}+8} \text{ and } g_{\tilde{n}+12} \quad (16)$$

appearing in the well-known *six*-dimensional F-theory [18, 19] compactified on an elliptic CY3 over a Hirzebruch surface $\mathbb{F}_{\tilde{n}}$, where the subscripts denote the degrees of the polynomials. More generally (in four dimensions), they are the sections of the line bundles

$$\begin{aligned} a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes \mathcal{N}), \\ a_{2,1} &\in \Gamma(\mathcal{L}^{-4} \otimes \mathcal{N}), \\ a_{3,2} &\in \Gamma(\mathcal{L}^{-3} \otimes \mathcal{N}), \\ a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes \mathcal{N}), \\ a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes \mathcal{N}) \end{aligned} \quad (17)$$

[3, 7, 16, 17], where \mathcal{L} is the anticanonical line bundle of the base of the heterotic threefold, and \mathcal{N} is the twisting line bundle for the vector bundle over this threefold. Γ denotes the space of sections. The divisors of these bundles must be effective. The most stringent constraint comes from $a_{1,0}$ ($\sim h_{\tilde{n}+2}$), which asserts that $\eta - 5c_1$ must be effective. Thus we have

$$10 \leq x \leq 24, \quad 10 + 5m \leq y \leq 24 + 12m. \quad (18)$$

For $SO(10)$, the section $a_{1,0}$ ($\sim h_{\tilde{n}+2}$) becomes zero, but instead $a_{2,1}$ ($\sim H_{\tilde{n}+4}$) must exist. Therefore $\eta - 4c_1$ must be effective. Similarly, $a_{2,1}$ becomes zero for E_6 , and then for $a_{3,2}$ ($\sim p_{\tilde{n}+6}$) to exist $\eta - 3c_1$ must be effective. Thus, in general

$$2n \leq x \leq 24, \quad (2 + m)n \leq y \leq 24 + 12m \quad (19)$$

for $n = 5, 4, 3$.

Before displaying the list of the solutions, a remark is in order about the compactness of the generalized Hirzebruch variety defined by the weights (9): We can assume $m \geq 0$ without

loss of generality since, if $m < 0$, we can consider the new scaling $\nu\mu^{-m}$ and interchange the roles of z' and w' . Similarly, if both of $(q, p) = (x - 12, y - 6m - 12)$ happen to be negative, we can consider the scalings $\nu\kappa^{-p}$ and $\mu\kappa^{-q}$ replace z with w and vice versa; this corresponds to consider the unbroken gauge group residing “at infinity” of the \mathbb{P}^1 fiber. If, however, only one of (q, p) happens to be negative, there is no way to define such a new scaling and the variety would become noncompact. Therefore we restrict ourselves to the solutions such that *both* of $(q, p) = (x - 12, y - 6m - 12)$ are non-negative, or *both* of $(q, p) = (x - 12, y - 6m - 12)$ are non-positive.

The complete list of the solutions satisfying the condition (I) or (II) or (I') or (II') in the range (19), and satisfying the condition in the remark above, is as follows:

$SU(5)$ models ($n = 5$)

- $\lambda = \frac{1}{2}$

$$\begin{aligned} (x, y) &= (12, 6m + 9) \quad (m \in \mathbb{Z}, m \geq 1), \\ &\quad (19, \frac{19}{2}m + 7) \quad (m \in 2\mathbb{Z}, m \geq 2), \\ &\quad (13, \frac{13}{2}m + \frac{17}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 7). \end{aligned}$$

- $\lambda = -\frac{1}{2}$

$$(x, y) = (16, 8m + 7) \quad (m \in \mathbb{Z}, m \geq 3).$$

- $\lambda = \frac{3}{2}$

$$(x, y) = (18, 9m + 7) \quad (m \in \mathbb{Z}, m \geq 2).$$

- $\lambda = -\frac{3}{2}$

$$\begin{aligned} (x, y) &= (11, \frac{11}{2}m + 9) \quad (m \in 2\mathbb{Z}, m \geq 2), \\ &\quad (13, \frac{13}{2}m + 8) \quad (m \in 2\mathbb{Z}, m \geq 8), \\ &\quad (17, \frac{17}{2}m + 7) \quad (m \in 2\mathbb{Z}, m \geq 2), \\ &\quad (21, \frac{21}{2}m + \frac{13}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 3). \end{aligned}$$

$SO(10)$ models ($n = 4$)

- $\lambda = \frac{1}{2}$

$$(x, y) = (23, \frac{23}{2}m + 5) \quad (m \in 2\mathbb{Z}, m \geq 2).$$

- $\lambda = -\frac{1}{2}$

$$(x, y) = (17, \frac{17}{2}m + 5) \quad (m \in 2\mathbb{Z}, m \geq 4).$$

- $\lambda = \frac{3}{2}$

$$(x, y) = (21, \frac{21}{2}m + 5) \quad (m \in 2\mathbb{Z}, m \geq 2).$$

- $\lambda = -\frac{3}{2}$

$$(x, y) = (9, \frac{9}{2}m + 7) \quad (m \in 2\mathbb{Z}, m \geq 2),$$

$$(19, \frac{19}{2}m + 5) \quad (m \in 2\mathbb{Z}, m \geq 2).$$

- $\lambda = 1$

$$(x, y) = (9, \frac{9}{2}m + \frac{15}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 1),$$

$$(11, \frac{11}{2}m + \frac{13}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 1),$$

- $\lambda = 3$

$$(x, y) = (15, \frac{15}{2}m + \frac{11}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 5),$$

- $\lambda = -1, -3$

No solution.

E_6 models ($n = 3$)

- $\lambda = \frac{1}{2}$

$$(x, y) = (6, 3m + 7) \quad (m \in \mathbb{Z}, m \geq 0),$$

$$(7, \frac{7}{2}m + 6) \quad (m \in 2\mathbb{Z}, m \geq 0),$$

$$(9, \frac{9}{2}m + 5) \quad (m \in 2\mathbb{Z}, m \geq 2),$$

$$(15, \frac{15}{2}m + 4) \quad (m \in 2\mathbb{Z}, m \geq 6),$$

$$(11, \frac{11}{2}m + \frac{9}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 1).$$

- $\lambda = -\frac{1}{2}$

$$\begin{aligned}(x, y) &= (9, \frac{9}{2}m + 4) \quad (m \in 2\mathbb{Z}, m \geq 2), \\ &\quad (7, \frac{7}{2}m + \frac{9}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 3), \\ &\quad (15, \frac{15}{2}m + \frac{7}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 7).\end{aligned}$$

- $\lambda = \frac{3}{2}$

$$\begin{aligned}(x, y) &= (8, 4m + 5) \quad (m \in \mathbb{Z}, m \geq 1), \\ &\quad (13, \frac{13}{2}m + 4) \quad (m \in 2\mathbb{Z}, m \geq 16), \\ &\quad (7, \frac{7}{2}m + \frac{11}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 1), \\ &\quad (23, \frac{23}{2}m + \frac{7}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 3).\end{aligned}$$

- $\lambda = -\frac{3}{2}$

$$\begin{aligned}(x, y) &= (7, \frac{7}{2}m + 5) \quad (m \in 2\mathbb{Z}, m \geq 2), \\ &\quad (11, \frac{11}{2}m + 4) \quad (m \in 2\mathbb{Z}, m \geq 2), \\ &\quad (19, \frac{19}{2}m + \frac{7}{2}) \quad (m \in 2\mathbb{Z} + 1, m \geq 3).\end{aligned}$$

One can verify that all the solutions listed above give precisely three generations while satisfying the condition (19). This is the main result of this letter.

In most cases, $m = 0$ is not allowed. The rare exceptions are the first two solutions for E_6 with $\lambda = \frac{1}{2}$.

We should note that the equation (1) has an obvious symmetry $\eta \rightarrow \tilde{\eta} = nc_1 - \eta$, so if some η satisfies (1), so does $\tilde{\eta}$, but this $\tilde{\eta}$ may or may not be a suitable divisor for which all the relevant divisors are effective. The \mathbb{F}_{2k+1} solution $\eta = (-3, -3k)$ for $SU(5)$ in ref.[8] has $\tilde{\eta} = (13, 13k + 15)$, which coincides with our solution $(13, \frac{13}{2}m + \frac{17}{2})$ with $m = 2k + 1$. Unlike our solution, however, neither $(-3, -3k)$ nor $(-3, -3k) - 5c_1$ is effective. The \mathbb{F}_{2k} solution $\eta = (3, 3(k - 2))$ for $SU(5)$ in ref.[8], or its $\tilde{\eta}$, is not contained in the above list because neither $\eta - 5c_1$ nor $\tilde{\eta} - 5c_1$ is effective. Also, the $SO(10)$ or E_6 solutions found in [8] or their $\tilde{\eta}$ do not coincide with any of our solutions since their relevant divisors are not effective, either. The “closest” is $\eta = (0, 1)$ for E_6 ($\lambda = -\frac{1}{2}$) which has $\tilde{\eta} = (6, 3m + 5)$ with a marginal value for x ($= 6$), but $y = 3m + 5$ is outside the range (19).

To summarize, we have shown a complete list of η classes leading to precisely three-generations in $E_8 \times E_8$ heterotic string theory on an elliptic Calabi-Yau over a Hirzebruch surface and its dual F-theory on an elliptic Calabi-Yau over a generalized Hirzebruch variety, where the divisors of the relevant line bundles needed for a smooth Weierstrass model are all effective. We hope they will be used as a concrete simple model to investigate fundamental questions in F-theory as we mentioned in the beginning of this letter.

We thank K. Kohri, K. Mohri and T. Tani for discussions. The work of S. M. is supported by Grant-in-Aid for Scientific Research (C) #25400285, (C) #16K05337 and (A) #26247042 from The Ministry of Education, Culture, Sports, Science and Technology of Japan.

-
- [1] M. I. Vysotsky, “*The rise and fall of the fourth quark-lepton generation,*” Proceedings, arXiv:1312.0474 [hep-ph].
 - [2] J. Lesgourgues and S. Pastor, New J. Phys. **16**, 065002 (2014) [arXiv:1404.1740 [hep-ph]].
 - [3] R. Donagi and M. Wijnholt, Adv. Theor. Math. Phys. **15**, 1237 (2011) [arXiv:0802.2969 [hep-th]].
 - [4] C. Beasley, J. J. Heckman and C. Vafa, JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]].
 - [5] C. Beasley, J. J. Heckman and C. Vafa, JHEP **0901**, 059 (2009) [arXiv:0806.0102 [hep-th]].
 - [6] R. Donagi and M. Wijnholt, Adv. Theor. Math. Phys. **15**, 1523 (2011) [arXiv:0808.2223 [hep-th]].
 - [7] R. Friedman, J. Morgan and E. Witten, Commun. Math. Phys. **187**, 679 (1997) [hep-th/9701162].
 - [8] G. Curio, Phys. Lett. B 435, 39 (1998) [hep-th/9803224].
 - [9] A. Klemm, B. Lian, S. S. Roan and S. T. Yau, Nucl. Phys. B **518** (1998) 515 [hep-th/9701023].
 - [10] K. Mohri, Int. J. Mod. Phys. A **14** (1999) 845 [hep-th/9701147].
 - [11] G. Rajesh, JHEP **9812**, 018 (1998) [hep-th/9811240].
 - [12] R. Donagi, Y. H. He, B. A. Ovrut and R. Reinbacher, Phys. Lett. B **618** (2005) 259 [hep-th/0409291].
 - [13] R. Donagi, Y. H. He, B. A. Ovrut and R. Reinbacher, JHEP **0506** (2005) 070 [hep-th/0411156].
 - [14] V. Bouchard and R. Donagi, Phys. Lett. B **633** (2006) 783 [hep-th/0512149].
 - [15] S. Mizoguchi, JHEP **1407**, 018 (2014) [arXiv:1403.7066 [hep-th]].

- [16] S. Mizoguchi and T. Tani, arXiv:1508.07423 [hep-th]. To appear in PTEP.
- [17] S. Mizoguchi and T. Tani, in preparation.
- [18] D. R. Morrison and C. Vafa, Nucl. Phys. B **473**, 74 (1996) [hep-th/9602114]; Nucl. Phys. B **476**, 437 (1996) [hep-th/9603161].
- [19] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, Nucl. Phys. B **481**, 215 (1996) [hep-th/9605200].